Analysis of Unsteady Viscous Flow Past an Airfoil: Part I—Theoretical Development

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An analysis is presented for the two-dimensional unsteady flow of a viscous incompressible fluid past a finite thickness airfoil at angle of attack. Using techniques commonly employed in ideal-fluid aerodynamic analyses, the airfoil surface is represented by bound-vortex singularities γ distributed over the outline of the airfoil. These coexist with the free vorticity ω in the boundary layer and wake. Using standard procedures, the integral equation for γ is cast into a Fredholm equation of the second kind. This differs from that found in inviscid analyses because of the contribution of ω to the induced velocities at the airfoil surface, thus coupling the two vorticity fields. A further coupling arises when the no-slip condition is enforced at the airfoil surface, the enforcement of which causes free vorticity to be produced. The production of free vorticity is modeled by equating the local instantaneous value of γ to the amount of free vorticity produced locally and impulsively at the airfoil surface. This free vorticity enters the fluid stream by diffusion, thus giving rise to the essential boundary condition which must be imposed on the transport equation governing the distribution of free vorticity in the fluid. The formulation is developed in detail for the case where the airfoil is impulsively set into translational motion. The analysis makes explicit use of the requirement that the total vorticity of the fluid must remain zero, thus removing any ambiguities in the solution for γ. This also insures that the pressure distribution on the airfoil remains single-valued. The numerical formulation and results for a particular airfoil are presented in Part II of the paper.

Nomenclature

$\hat{A}$	= complementary solution for the auxiliary bound vorticity
h	= $1+\eta/R$, scale factor in body-oriented coordinates, Fig. 1
K	= kernel function in velocity induction law
L	= chord length of the airfoil
P	= static pressure
R	= radius of curvature of the airfoil surface
r	= linear distance between two points
Re	= Reynolds number, UL/ν
t	= time
U	= velocity of the undisturbed onset flow relative to the airfoil
u	= velocity component in the ξ direction
V	= volume
v	= velocity component in the η direction
x	= Cartesian coordinate measured along the chord from the center
y	= Cartesian coordinate perpendicular to the chord
α	= angle of attack
γ	= bound vorticity distribution
$\hat{\gamma}$	= auxiliary function, Eq. (7b)
η	= body-oriented coordinate measured perpendicular to airfoil surface
θ	= slope angle of airfoil surface relative to the chord
ν	= kinematic viscosity
ξ	= body-oriented coordinate measured along the airfoil surface
ρ	= fluid density
τ	= shear stress
ϕ,ψ	= polar angles measured from the real axis in the transformed (circle) plane
Ω	= total vorticity
ω	= free vorticity

Received July 12, 1976; revision received Aug. 24, 1977.

Index categories: Viscous Nonboundary-Layer Flows; Nonsteady Aerodynamics.

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Introduction

SINGULARITY methods have played an important role in ideal-fluid aerodynamic analyses. The familiar theories employ either source or vortex singularities, and Hess and Smith[1] have given an excellent accounting of the wide variety of two- and three-dimensional flows which can be treated by these methods. Although most of their attention is given to source singularities, some mention is made of the vortex methods. Among these, the works by Prager[2] and Martensen[3] are particularly noteworthy. Extensions to nonlinear unsteady flows have also been made by Djojodihardjo and Widnall.[4]

In all of these representative works, the modeling of potential flows is stressed. Nevertheless, there is no inherent limitation which precludes singularity distributions from use in unsteady viscous-flow analyses. That is to say, their role is purely kinematic, and they must in all cases be supplemented with suitable dynamical constraints. Sears[5] has recently emphasized this point in his discussion of airfoils with boundary-layer separation. He argues convincingly that the bound-vortex singularities used in classical theories do actually model the airfoil surface plus viscous boundary layers. This leads to a general interpretation of the Kutta-Joukowski condition, when the dynamic process of vortex shedding at the trailing edge is related to the rate of change of circulation. This serves to remind us that the so-called inviscid-fluid models do nothing more than represent the limiting case of vanishingly small viscosity and not the flow of a truly inviscid fluid.

When the fluid viscosity is not small, a distribution of vortex singularities may still be used to represent instantaneously the surface of the body, but this becomes distinct from the viscous layers which surround the body and form the wake. Special attention must be given to the vorticity dynamics over the entire surface of the airfoil, not just at the trailing edge. This involves the proper accounting for the production of vorticity at the solid surface and its subsequent transport throughout the finite-thickness boundary layer and wake. Furthermore, this must always be carried out in such a way that the total circulation is conserved at all times. A numerical technique based on these ideas has already been used by Schmall and Kinney[6] to model the two-dimensional unsteady viscous flow past a finite lifting plate.

The advantage of singularity methods in potential-flow analyses is that the vortex-distributions are governed by integral equations over the airfoil and wake surfaces. Thus, even when numerical solutions are required, calculations need be performed only over this limited region, and not the entire extent of the disturbed flow. This advantage carries over to fully viscous-flow analyses as well, in which case calculations need be carried out only in the regions of nonzero vorticity. The above cited analysis,[6] as well as the integro-differential formulation of Wu and Thompson,[7] exploit this advantage. Somewhat analogous approaches by Chorin,[8] Panikker and Lavan,[9] and Bratanow and Ecer[10] have this feature.

In the present paper, an analysis is made of the two-dimensional unsteady viscous flow over an airfoil impulsively started from rest, and which has some specified incidence angle relative to its chord. The airfoil moves in translation only. Part I of this paper is devoted to the theoretical development. The companion paper, Part II, presents the numerical formulation and results.

The present paper is an extension of work presented earlier by the same authors.[11] It is, in a broad sense, an exact formulation of the notions already articulated by Sears,[5] and is therefore in contrast to the conventional stream-function formulations which are already in wide use. For a discussion of these, the reader is referred to the excellent work of Lugt and Haussling[12] and Mehta and Lavan.[13]

Description of Theoretical Approach

The theoretical foundations adopted here for analyzing the incompressible flowfield are quite general and can be found in many elementary textbooks on fluid mechanics[14] and aerodynamics.[15] The specific modeling of the unsteady viscous-flow development is based on the procedure originally given by Lighthill.[16]

Basically, the analysis is decomposed into those aspects which are purely kinematic and those which are purely dynamic. The kinematic description is presented first and is based on the law of induced velocities (Biot-Savart law). This allows a one-to-one relationship to be established between the vorticity and velocity fields in the fluid. The dynamic description concerns the production and transport of vorticity and will be taken up separately.

In order to apply the velocity induction law, we suppose the airfoil to be removed and its place taken by fluid of the same density as the undisturbed surroundings and moving as an ensemble with the velocity $-U$ of the airfoil. That is, the airfoil moves from right to left. The surface of the airfoil is replaced by a vortex sheet of vorticity γ. This vortex distribution will be called the "bound vorticity" since it always occupies the location of the airfoil surface. In general, the fluid surrounding the airfoil contains "free vorticity" ω, and at any instant in time the application of the velocity induction law gives the actual velocity of the fluid at any point in the field. Furthermore, it can be shown that for a given distribution of ω, a distribution of γ can always be found such that all the fluid interior to the airfoil has the ensemble velocity $-U$. In fact, the distribution of γ is not uniquely determined by this condition alone.

We now superpose on the whole system a velocity $+U$ such that the fluid appears to stream past the airfoil from left to right, and the fluid in the interior is at rest. At infinity, the fluid is no longer at rest but has velocity U. Thus, one of the conditions for the application of the induction law is violated. Nevertheless, superposition of the same velocity U on the entire system does not alter the distributions of γ and ω, since these depend only on spatial differences of velocity. In this moving coordinate system, then, we speak of the velocity field induced by γ and ω as the "perturbation velocity," and the fluid velocity is now given by U plus the vector contributions due to γ and ω, as obtained from the induction law.

The velocity field just described may not in general satisfy the surface adherence condition so necessary to a viscous-flow analysis. In particular, the fluid may appear at any instant in time to have a nonzero tangential component on the exterior surface of the airfoil. In fact, this "apparent" slip velocity is numerically equal to $-\gamma$ at each point on the airfoil. As explained by Lighthill,[16] this apparent slip velocity must be reduced to zero by the production of a precise quantity of free vorticity at each point on the airfoil. This vorticity then enters the fluid and is distributed throughout by convection and diffusion. The production and redistribution of the free vorticity is governed by dynamic equations. Although not explicitly mentioned by Lighthill, a global conservation principle must also be enforced which requires that the total vorticity (or circulation about the airfoil) be conserved for all time.

The main task, then, is to determine simultaneously the bound and free vorticity distributions, at each instant in time, such that 1) the viscous adherence condition is satisfied at the airfoil surface, 2) the free vorticity satisfies the dynamic equation describing its production and transport, and 3) the total vorticity of the field is conserved. Once the distributions for γ and ω are found, the entire flowfield is uniquely determined, and it must in fact satisfy the full unsteady Navier-Stokes equations.

We feel that there are advantages to be realized by isolating the kinematic and dynamic aspects of the flowfield development with time. Specifically, this should yield an accurate representation for the flow near the airfoil (as it does in ideal-fluid calculations), thereby providing more accurate surface-force predictions than are possible with conventional methods based on the stream function. Most important, however, the kinematic and dynamic descriptions based on the vorticity carry over directly to three-dimensional flows, an extension which is not possible with stream-function formulations. In summary, then, the present approach appears to have great potential for treating a wide class of external viscous-flow problems.

Analysis

Vorticity Kinematics

A schematic of the airfoil is shown in Fig. 1. A system of body-oriented coordinates is chosen such that ξ is measured from the leading edge around the contour of the airfoil, and then along the extension of the chord-line behind the trailing edge; η is measured normal to ξ. A Cartesian system is also used with origin at the midcord. The subscript S denotes quantities which are constrained to the airfoil surface. Thus $y_S(x_S)$ is the known representation for the outline of the airfoil.

The velocity components in the ξ and η directions are denoted by u and v, respectively. The onset flow has velocity

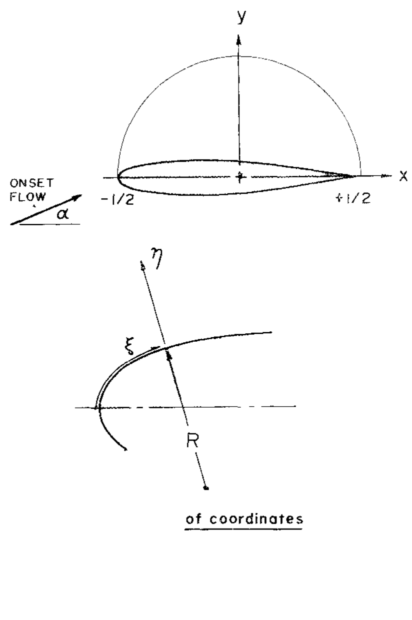

Fig. 1 System of coordinates.

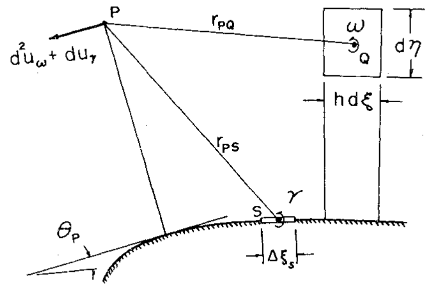

Fig. 2 Vorticity contributions to induced velocity field.

magnitude U and approaches the airfoil as shown, the angle of attack being denoted by α. All variables are rendered nondimensional by U, the chord length L, and the kinematic viscosity of the fluid ν.

An expression for the velocity component in the ξ direction at any point P in the fluid will now be developed. Consider an infinitesimal region of the fluid with free vorticity ω and incremental area $h\mathrm{d}\xi\mathrm{d}\eta$. This is shown in Fig. 2. The scale factor h is equal to $1+\eta/R$, where R is the radius of curvature of the airfoil. An incremental segment of the airfoil surface, with bound vorticity γ, is also shown in Fig. 2.

The infinitesimal contribution to the u-velocity at P from these two vorticity distributions can now be written as

$$\mathrm{d}^2u_\omega(P) = -\frac{1}{2\pi}\left[\frac{(y_P-y_Q)\cos\theta_P-(x_P-x_Q)\sin\theta_P}{r_{PQ}^2}\right](\omega h\mathrm{d}\xi\mathrm{d}\eta)_Q \quad (1a)$$

$$\mathrm{d}u_\gamma(P) = -\frac{1}{2\pi}\left[\frac{(y_P-y_S)\cos\theta_P-(x_P-x_S)\sin\theta_P}{r_{PS}^2}\right](\gamma\mathrm{d}\xi)_S \quad (1b)$$

where θ_P is the angle between the chord and the tangent to the airfoil at ξ_P. Also, positive values are assigned to ω and γ when these have a counterclockwise sense of rotation. Note that only the scalar magntiudes need be considered for this two-dimensional problem.

The perturbation velocity at P is now obtained by integrating over the entire vorticity field all the individual contributions from Eqs. (1). After this is added to the contribution from the onset flow, one obtains

$$u(P)=\cos(\alpha-\theta_P)+\int_A \mathrm{d}^2u_\omega+\oint_C \mathrm{d}u_\gamma \quad (2)$$

The transverse velocity v can be calculated from a set of equations similar to those given in Eqs. (1). However, it is more expedient to obtain this from the continuity equation, once the u-velocities are known. The continuity equation in our system can be written

$$\frac{\partial u}{\partial \xi}+\frac{\partial}{\partial \eta}(hv)=0 \quad (3)$$

In order to find an equation which governs the bound vorticity, Eq. (2) must be specialized for a point P on the airfoil surface, and the fluid velocity interior to the airfoil surface must be set to zero. The derivation is quite straightforward, but it is important to keep in mind that γ represents a discontinuity in the tangential velocity component. Therefore, care must be taken to distinguish between the velocity in the interior and exterior regions of the airfoil at P.

Let (+) denote the velocity at P on the exterior surface of the airfoil and (−) denote the velocity on the interior. The integration of Eq. (1a) over the entire free vorticity field can be carried out for known ω, and we denote this quantity by $u_\omega(P)$. Continuing, the integration of Eq. (1b) over the airfoil contour is now written in two parts. If $\Delta\xi_P$ denotes the incremental arc length at P, one part contains the integral from $-\Delta\xi_P/2$ to $+\Delta\xi_P/2$, and the second part the integral over $C-\Delta\xi$, where C is the contour length. It is well known from classical airfoil theory that as $\Delta\xi_P\to 0$, the integral over $\Delta\xi_P$ approaches $\mp\gamma_P/2$, depending on whether the point P is on the exterior or interior side of the airfoil surface. One thus obtains

$$u_S^+(P)=\cos(\alpha-\theta_P)+u_\omega(P)-\frac{1}{2\pi}\oint_C K\gamma\mathrm{d}\xi-\frac{\gamma_P}{2} \quad (4a)$$

$$u_S^-(P)=\cos(\alpha-\theta_P)+u_\omega(P)-\frac{1}{2\pi}\oint_C K\gamma\mathrm{d}\xi+\frac{\gamma_P}{2} \quad (4b)$$

from which it is seen that $u_S^+(P)-u_S^-(P)=-\gamma_P$. Also, K is an abbreviation for the term in brackets in Eq. (1b).

The velocity of the fluid interior to the airfoil is now equated to zero. That is, $u_S^-(P)=0$, and one obtains the following integral equation for γ

$$\gamma_P-\frac{1}{\pi}\oint_C K\gamma\mathrm{d}\xi=-2[\cos(\alpha-\theta_P)+u_\omega(P)] \quad (5)$$

It will now be recognized that Eq. (5) is a Fredholm integral equation of the second kind. Except for the term $u_\omega(P)$, it is identical to that derived by Prager[2] and subsequently studied by Martensen.[3] Thus the solution properties are well known. For our development, it is only necessary to keep in mind that the term $u_\omega(P)$ does couple Eq. (5) to the solution for the free vorticity field. However, it only modifies the nonhomogeneous term, a quantity which is considered known in any event. We now proceed with the solution of Eq. (5) for γ. This requires only an application of known techniques developed for ideal-fluid analyses.[3] Once γ is found, the apparent slip velocity on the external surface of the airfoil follows immediately, since it is equal to $-\gamma$. This is a direct result of Eqs. (4), with $u_S^-(P)=0$.

It is convenient to recast Eq. (5) into parametric form using a complex conformal representation. The cylindrical section of the airfoil in the physical plane is considered to be mapped onto a circle in the transformed plane. Then for every point (x_S,y_S) on the airfoil, there is a corresponding point on the circle with some polar angle measured from the positive real axis. Let the polar angle ϕ denote the location of one such point on the circle and ψ that of any other surface point. Then

$$K(\phi,\psi)=\frac{[y_S(\phi)-y_S(\psi)]\cos\theta(\phi)-[x_S(\phi)-x_S(\psi)]\sin\theta(\phi)}{[x_S(\phi)-x_S(\psi)]^2+[y_S(\phi)-y_S(\psi)]^2} \quad (6a)$$

and

$$\mathrm{d}\xi=\sqrt{\mathrm{d}x_S^2+\mathrm{d}y_S^2}=\sqrt{\dot{x}_S(\psi)^2+\dot{y}_S(\psi)^2}\,\mathrm{d}\psi \quad (6b)$$

where $\dot{x}_S(\psi)\equiv \mathrm{d}x_S/\mathrm{d}\psi$, and so forth. Also note that $\cos\theta$ and $\sin\theta$ are known functions of ϕ and are given by

$$\cos\theta(\phi)=-\frac{\dot{x}_S(\phi)}{\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2}} \quad (6c)$$

$$\sin\theta(\phi)=-\frac{\dot{y}_S(\phi)}{\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2}} \quad (6d)$$

The negative signs appear since increasing values of ϕ imply decreasing values of x, and θ is positive if $\dot{y}_S(\phi)$ is negative. After Eqs. (6a-d) are substituted into Eq. (5) and the result is multiplied through by $\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2}$, the following integral equation is obtained

$$\hat{\gamma}(\phi)-\frac{1}{\pi}\int_0^{2\pi}\hat{K}(\phi,\psi)\hat{\gamma}(\psi)\mathrm{d}\psi=-2\{\cos[\alpha-\theta(\phi)]+u_\omega(\phi)\}\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2} \tag{7a}$$

where

$$\hat{\gamma}(\phi)=\gamma(\phi)\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2} \tag{7b}$$

and

$$\hat{K}(\phi,\psi)=K(\phi,\psi)\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2} \tag{7c}$$

As noted by Prager,[2] the solution to Eq. (7a) is composed of two parts. These are the particular solution to the complete equation and the complementary solution to the homogeneous equation. The particular solution can be obtained in a straightforward manner, but because of the term u_ω, numerical methods have to be employed. This is taken up in Part II.

The complementary solution corresponds to the case of pure circulation about the airfoil,[2] and although it was not derived by Prager[2] or Martensen,[3] it can be obtained with the help of the complex conformal representation already adopted. Let $\gamma_C(\phi)$ be the complementary solution to Eq. (5) evaluated at some point on the airfoil in the physical z plane. Recall that except for a possible sign change, this is equal to the surface slip velocity. Clearly, when the corresponding surface velocity is obtained in the transformed (circle) ζ plane, it must be a constant, say $\hat{C}$. Thus, we can write $\gamma_C(\phi)\cdot|\mathrm{d}z/\mathrm{d}\zeta|=\hat{C}$. In our convention, $|\mathrm{d}z|=\mathrm{d}\xi$ and $|\mathrm{d}\zeta|=R\mathrm{d}\phi$, where R is the radius of the circle in the ζ plane. It thus follows that $\gamma_C(\phi)\cdot(\mathrm{d}\xi/\mathrm{d}\phi)=\hat{A}$, where $\hat{A}$ is a new constant (i.e., the product of $\hat{R}$ and $\hat{C}$), and $\mathrm{d}\xi/\mathrm{d}\phi$ is given by Eq. (6b) with ψ replaced by ϕ. In view of Eq. (7b), we therefore conclude that the complementary solution to Eq. (7a) must be a constant.

In summary, then, the solution for the bound vorticity, γ, can be written

$$\gamma(\phi)=\frac{\hat{\gamma}_{\mathrm{part}}(\phi)+\hat{A}}{\sqrt{\dot{x}_S(\phi)^2+\dot{y}_S(\phi)^2}} \tag{8}$$

where $\hat{\gamma}_{\mathrm{part}}$ and $\hat{A}$ are the particular and complementary solutions, respectively.

The solution for γ is not completely specified until $\hat{A}$ is evaluated. This requires consideration of the vorticity dynamics at the airfoil surface and is taken up in the next section. Before turning to this, however, it should be noted that considerable use has been made of the Martensen method to calculate potential flows around airfoils with sharp trailing edges (see Ref. 17). However, some caution is required near the trailing edge ($\phi=0$ in the circle plane) since the denominator of Eq. (8) goes to zero. Nevertheless, in this region $\gamma\mathrm{d}\xi$ remains finite, even at the trailing edge. This is important to keep in mind, since it is not the value of γ itself which is of interest, but rather the product $\gamma\mathrm{d}\xi$. Of course, if the classical Kutta-Joukowski condition is used to force γ equal to zero at the trailing edge, then one finds directly that $\hat{A}$ is equal to the negative value of $\hat{\gamma}_{\mathrm{part}}$ at the trailing edge.

Vorticity Dynamics

The bound vorticity distribution is coupled to the free vorticity field through the term u_ω in Eq. (5). That is, the distribution of ω throughout the fluid must be known before Eq. (1a) can be integrated. This distribution of vorticity is governed by the usual unsteady transport equation, which for our system of coordinates can be written:

$$\frac{\partial\omega}{\partial t}+\frac{1}{h}\left[\frac{\partial}{\partial\xi}(u\omega)+\frac{\partial}{\partial\eta}(hv\omega)\right]=\frac{1}{hRe}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{h}\frac{\partial\omega}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(h\frac{\partial\omega}{\partial\eta}\right)\right] \tag{9}$$

In the foregoing, all variables have been rendered dimensionless with L, U, and ν.

Initially the free vorticity is everywhere zero, and since Eq. (9) contains no volume source terms, a mechanism must be provided for the production of the free vorticity which eventually permeates the viscous fluid. It is known that the mechanism is provided when the viscous-adherence condition is enforced. In the present study, the satisfaction of the adherence condition is achieved using a concept which was originally proposed by Lighthill[16] and which has been used previously by Schmall and Kinney.[6] This is an important feature of the analysis, and one which requires further discussion.

To aid the following description, it is necessary to distinguish between the several "vorticity" quantities which up to now have been treated rather loosely. Also, all quantities will be taken to be dimensional. The term *total vorticity* will be used to denote the amount of vorticity in a given volume of fluid (area multiplied by a unit length perpendicular to the plane of flow). We give this the symbol Ω. The free vorticity ω is now interpreted as being the *total vorticity per unit volume*. In this convention, Ω has dimensions of length^3/time, and ω has the usual dimensions of 1/time. Clearly, $\mathrm{d}\Omega=\omega\mathrm{d}V$, where $\mathrm{d}V$ is an elemental volume. For two-dimensional flows, the quantity $-\nu(\partial\omega/\partial\eta)_s$ represents the *flux of total vorticity* from the airfoil surface into the fluid. It has units of *total vorticity per unit area per unit time*, or simply length/time^2. This term is analogous to the rate expression describing the molecular diffusion of heat as the result of a temperature gradient. We thus envision that *total vorticity* diffuses as a result of a gradient in *vorticity* (*temperature* does not diffuse as a result of a gradient in *temperature*). Finally, the bound vorticity γ is identified as the *amount of total vorticity per unit surface area of the airfoil* which is produced impulsively and diffuses into the fluid over a given time interval. That is, it has units of length/time, or velocity. According to the adopted convention, then, one can write

$$-\nu\int_t^{t+\Delta t}\left(\frac{\partial\omega}{\partial\eta}\right)_S\mathrm{d}t'=\gamma(x_S,y_S,t) \tag{10}$$

The above expression applies for $t=0^+$ as well, and is therefore applicable immediately after the airfoil is accelerated impulsively. It is an explicit statement of the process of vorticity production described only verbally by Lighthill,[16] and it is the essential boundary condition to be used in conjunction with Eq. (9).

Immediately after the impulsive start, γ is nonzero over the airfoil surface. In fact, it corresponds to the potential-flow slip velocity for zero circulation.† In order to reduce the slip velocity to zero, we require that immediately and impulsively, *total vorticity* γ per unit area be produced at each point of the airfoil. However, as Eq. (10) implies, the new vorticity is allowed to diffuse into the fluid over a finite time span Δt. Although there are practical considerations which must be observed when specifying Δt (to be discussed at a later point), Eq. (10) implies that it is the amount of total vorticity

†The only difference between the surface slip velocity and γ is in the algebraic sign. If the fluid slips in the direction from the leading to trailing edges, then γ is negative on the upper surface and positive on the lower surface of the airfoil.

produced which is important, and not $(\partial\omega/\partial\eta)_s$ or Δt. Stated another way, it is the right-hand side of Eq. (10) which is known. Thus, the smaller Δt becomes, the larger must be $(\partial\omega/\partial\eta)_s$ in order that the integrated result remain unchanged.

As the solution for the vorticity is advanced in time, the viscous adherence condition is actually enforced at the beginning of each time increment. At the end of a time step, a new distribution for ω is obtained through the integration of Eq. (9), taking into account the new vorticity which has entered (or left) the fluid through the airfoil surface, as represented by Eq. (10). One must then calculate a new distribution of u_ω over the airfoil surface, and ultimately a new distribution of γ. At points where γ is not zero, new free vorticity must be produced and so on and so forth.

Up to this point, no use has been made of the principle of conservation of total vorticity, nor has the solution for γ been completely specified. The latter involves the application of the former, a step which can now be carried out.

For an airfoil started from rest and moving in pure translation, the total vorticity entering the fluid from the entire airfoil surface, over each time increment, must be zero. This follows from the general principles articulated by Sears.[5] Therefore, the following additional constraint must be imposed:

$$\oint_C \left[-\nu \int_t^{t+\Delta t} \left(\frac{\partial\omega}{\partial\eta}\right)_S \mathrm{d}t' \right] \mathrm{d}\xi = 0 \tag{11}$$

In view of Eq. (10), this requires that the integral of the bound vorticity over the airfoil contour also vanish. From this conclusion plus Eqs. (8) and (6b), one can write

$$\int_0^{2\pi} (\hat{\gamma}_{\text{part}} + \hat{A})\,\mathrm{d}\psi = \int_0^{2\pi} \hat{\gamma}_{\text{part}}\,\mathrm{d}\psi + 2\pi\hat{A} = 0 \tag{12}$$

where use has been made of the fact that $\hat{A}$ is a constant. After solving for $\hat{A}$ from Eq. (12) and substituting back into Eq. (8), the complete solution for γ becomes

$$\gamma(\phi) = \frac{\hat{\gamma}_{\text{part}}(\phi) - \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \hat{\gamma}_{\text{part}}(\psi)\,\mathrm{d}\psi}{\sqrt{\dot{x}_S(\phi)^2 + \dot{y}_S(\phi)^2}} \tag{13}$$

It remains to specify a practical criterion for the choice of the time increment Δt which appears in Eq. (10). This can be obtained by examining an approximation to the distribution of new free vorticity immediately adjacent to the airfoil surface.

Equation (9) is first written for the region adjacent to the airfoil, with $v = \partial\omega/\partial\xi = 0$, and $h = 1$. This equation can then be solved subject to the dimensionless form of the boundary condition corresponding to Eq. (10). One obtains for the new free vorticity

$$\omega^*(\xi,\eta,t+\Delta t) = 2\gamma(\xi,t)\frac{Re}{\Delta t}\,\mathrm{ierfc}\left(\sqrt{\frac{Re}{\Delta t}}\frac{\eta}{2}\right) \tag{14}$$

where it has been assumed that the new vorticity (denoted by ω^*) is produced at a uniform rate over the time interval Δt. That is, the boundary condition becomes $-(1/Re)(\partial\omega/\partial\eta)_s = \gamma(\xi,t)/\Delta t$. Finally, ierfc is the integral of the complimentary error function. In viewing Eq. (14), it is important to keep in mind that ω^* is not the actual vorticity in the fluid. To obtain this, ω^* must be added to that which is already in the fluid and which is due to events occurring prior to time t.

The distribution for ω^* from Eq. (14) falls off rapidly with η, and is less than 1% of its wall value for $(Re/\Delta t)^{1/2}\eta/2$ greater than 1.6. Following Lighthill,[16] ω^* is assumed to be confined to a layer of thickness $\eta = \delta$, this being the extent of the vortex sheet which during time Δt has been broadened by diffusion. For fixed δ, then $\Delta t \lesssim Re\delta^2/10$. The selection of δ depends on numerical considerations and will be taken up in Part II.

Evaluation of Surface Forces

The calculation of the shear and pressure forces on the airfoil surface requires that the vorticity and its derivative with respect to η be evaluated at $\eta = 0$. In particular, if the shear and pressure stresses are rendered dimensionless by $\rho U^2/2$, one obtains

$$\tau_S = -\frac{2}{Re}\omega_S \tag{15a}$$

$$\left(\frac{\partial P}{\partial\xi}\right)_S = -\frac{2}{Re}\left(\frac{\partial\omega}{\partial\eta}\right)_S \tag{15b}$$

Since ω_s and $(\partial\omega/\partial\eta)_s$ are not known explicitly, their values must be inferred indirectly. This matter will be discussed more fully in Part II, when the numerical methods are presented. Nevertheless, some comments can be made here about the evaluation of Eq. (15b).

If we assume, as was previously done, that the rate of vorticity production is uniform over the time t to $t+\Delta t$, then one can write

$$\left(\frac{\partial P}{\partial\xi}\right)_S = 2\frac{\gamma(\xi,t)}{\Delta t} \tag{16}$$

where the pressure gradient can be considered to be evaluated at any time within the interval in question. When this quantity is then integrated around the closed contour of the airfoil, the result must be zero, as can be seen from the arguments leading to Eq. (12). Thus the pressure is inherently a single-valued function, a feature which is very desirable and which is not always achieved when stream-function formulations are used (see, e.g., Ref. 13).

Concluding Remarks

The analytical development presented here is in every respect an extension and generalization of the approach already used in Refs. 6 and 11. Still, a considerable amount of discussion has been devoted to the vorticity dynamics and kinematics at the airfoil surface. The importance of correctly modeling the vorticity production at the airfoil surface, in such a way that the total vorticity of the fluid is conserved, cannot be overstressed.

This point has received further emphasis in a very recent paper by Wu,[18] which appeared while the present paper was under review. The similarities between Wu's approach and that of the present paper are striking, considering that each approach has evolved quite independently. Interestly, the statement of vorticity conservation used herein [Eq. (11)], is identical to Eq. (10) of Wu's paper, when specialized to a body undergoing no angular accelerations.

Further similarities arise in the treatment of the bound vorticity, although Wu refers to this as the sheet vorticity on the surface. Nevertheless, Wu's Eq. (17) is equivalent to Eq. (5) of the present work. However, a slight departure arises when the connection is made between the sheet vorticity and the boundary condition to be used in solving the vorticity transport equation. Wu distributes the sheet vorticity over a fluid layer of finite thickness and then equates the vorticity thus obtained to the wall value (at the new time level $t+\Delta t$) needed in the solution of Eq. (9). It will be recalled that in the present work, the sheet strength is equated to the amount of free vorticity produced instantaneously at time t. This is subsequently allowed to diffuse into the fluid over a finite time interval Δt. Clearly, these two approaches are quite similar and may actually lead to indistinguishable results when implemented into a numerical procedure.

In Part II of this paper, the numerical formulation is developed. Results obtained for a specific airfoil geometry are presented, thereby demonstrating the utility of the method.

Acknowledgment

This work was supported by the National Science Foundation under Grant No. ENG73-03855 A01. The revised manuscript was prepared while the first author was on leave to the DFVLR-AVA, Institut für Aeroelastik, Göttingen, Germany. The financial support of the Alexander von Humboldt Foundation, Bonn, Germany, is gratefully acknowledged.

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